

Identical particles and entanglement^{*}

GianCarlo Ghirardi[†]

Department of Theoretical Physics of the University of Trieste, and
International Centre for Theoretical Physics “Abdus Salam”, and
Istituto Nazionale di Fisica Nucleare, Sezione di Trieste, Trieste, Italy
and

Luca Marinatto[‡]

Department of Theoretical Physics of the University of Trieste, and
Istituto Nazionale di Fisica Nucleare, Sezione di Trieste, Trieste, Italy

Abstract

We review two general criteria for deciding whether a pure bipartite quantum state describing a system of two identical particles is entangled or not. The first one considers the possibility of attributing a complete set of objective properties to each particle belonging to the composed system, while the second is based *both* on the consideration of the Slater-Schmidt number of the fermionic and bosonic analog of the Schmidt decomposition *and* on the evaluation of the von Neumann entropy of the one-particle reduced statistical operators.

1 Introduction

According to Schrödinger quantum entanglement represents “*the characteristic trait of Quantum Mechanics, the one that enforces its entire departure from classical lines of thoughts*” [1] due to its peculiar features. Nowadays entanglement is regarded as the most valuable resource in quantum information and quantum computation theory and therefore an extensive investigation of its features both from the theoretical and from the practical point of view is going on. In fact the possibility of successfully implementing teleportation processes [2], of devising efficient quantum algorithms outperforming the classical ones in solving certain computational problems [3] and of exhibiting secure cryptographical protocols [4], are grounded on the striking physical properties of entangled states. However, despite the fact that almost all the physical realization of the above-mentioned processes involve the use of identical particles, the very notion of entanglement in systems composed of indistinguishable elementary constituents seems to be lacking both of a satisfactory theoretical formalization and of a clear physical understanding. In fact the symmetrization postulate forces the physical systems composed of identical fermions and bosons to be described by states possessing definite symmetry properties under the permutation of the particle indices. As a consequence, these states generally display (i) a non-factorized form, (ii) a Schmidt number greater than 1, and (iii) a von Neumann entropy of the reduced single-particle

^{*}Work supported in part by Istituto Nazionale di Fisica Nucleare, Sezione di Trieste, Italy

[†]e-mail: ghirardi@ts.infn.it

[‡]e-mail: marinatto@ts.infn.it

statistical operator greater than 0¹. Therefore, if one would apply to systems composed of identical constituents the criteria which are commonly used for distinguishable particles, one would be naturally, but mistakenly, led to the conclusion that non-entangled states of identical fermions and bosons cannot exist.

In order to clarify about this common misunderstanding, which originates from confusing the unavoidable correlations due to the indistinguishable nature of the particles with the genuine correlations due to the entanglement, we will briefly and schematically review two equivalent criteria we have devised for deciding whether a given state is entangled or not. While the first is based on the possibility of attributing a *complete set of objective properties* to each component particle of the composed quantum system [5, 6], the second is based *both* on the consideration of the Slater-Schmidt number of the fermionic and bosonic analog of the Schmidt decomposition *and* on the evaluation of the von Neumann entropy of the one-particle reduced statistical operator [7].

2 Entanglement for distinguishable particles

Let us start by recalling the basic features of non-entangled (pure) state vectors describing composite systems of two distinguishable particles. Given a bipartite state $|\psi(1,2)\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$, the following three equivalent criteria represent necessary and sufficient conditions in order that the state can be considered as **non-entangled**:

1. $|\psi(1,2)\rangle$ is factorized, i.e., there exist two single-particle states $|\phi\rangle_1 \in \mathcal{H}_1$ and $|\chi\rangle_2 \in \mathcal{H}_2$ such that $|\psi(1,2)\rangle = |\phi\rangle_1 \otimes |\chi\rangle_2$. In this situation a well-defined state vector is assigned to each component subsystem and, since such states are simultaneous eigenstates of a complete set of commuting observables, it is possible to predict with certainty the measurement outcomes of this set of operators. These outcomes are exactly the objective properties which can be legitimately thought as possessed by each particle.
2. The Schmidt number of $|\psi(1,2)\rangle$, that is, the number of non-zero coefficients appearing in the Schmidt decomposition of the state, equals 1.
3. Given the reduced statistical operator $\rho^{(i)}$ of one of the two subsystems ($i = 1, 2$), its von Neumann entropy $S(\rho^{(i)}) = -\text{Tr}[\rho^{(i)} \log \rho^{(i)}]$ equals 0². Since the von Neumann entropy measures the uncertainty about the quantum state to attribute to a physical system, its value being null mirrors the fact that, in this situation, there is no uncertainty at all concerning the properties of each subsystem.

On the contrary, a bipartite quantum system is described by an **entangled** state $|\psi(1,2)\rangle$ if and only if one of the three following equivalent conditions holds true: (i) the state is not factorizable; (ii) the Schmidt number of the state is strictly greater than 1; (iii) the von Neumann entropy of both reduced statistical operators is strictly positive.

In this situation definite state vectors cannot be associated with each constituent and therefore we cannot claim that they possess objectively a complete set of properties whatsoever.

¹The only exception to these statements is represented by a system of bosons, each described by the same state vector.

²For our convenience, the log function is intended to be in base 2 rather than in the natural base e .

Accordingly, a strictly positive value of the von Neumann entropy reflects this uncertainty concerning the state of the particles.

3 Two identical particles

Let us now pass to analyze the case of interest, that is composite systems with identical constituents. In this situation the symmetrization postulate constraints the state associated with the system to be totally antisymmetric or symmetric under permutation of the two identical fermions or bosons respectively. Consequently the state is no longer factorized, its Schmidt decomposition involves generally more than one term and the von Neumann entropy of its reduced single-particle statistical operators is strictly positive³. It is then evident that a bipartite state vector describing two indistinguishable particles must (almost always) be considered entangled according to the criteria we have outlined in the previous section. The conclusion we have reached is clearly not correct and the origin of the problem resides in not having taken properly into account the role played by the unavoidable correlations which are due to the indistinguishability of the particles involved, correlations which are not connected with those arising from a genuine entanglement. In order to tackle this problem in the correct way, let us begin by sticking to the idea that the physically most interesting and fundamental feature of non-entangled states is that *both constituents possess a complete set of objective properties*.

In Refs. [5, 6] we have taken precisely this attitude, and the following definitions formalizing this point have been given:

Definition 3.1 *The identical constituents \mathcal{S}_1 and \mathcal{S}_2 of a composite quantum system $\mathcal{S} = \mathcal{S}_1 + \mathcal{S}_2$ are **non-entangled** when both constituents possess a complete set of properties.*

Definition 3.2 *Given a composite quantum system $\mathcal{S} = \mathcal{S}_1 + \mathcal{S}_2$ of two identical particles described by the normalized state vector $|\psi(1, 2)\rangle$, we will say that one of the constituents possesses a complete set of properties iff there exists a one-dimensional projection operator P , defined on the single particle Hilbert space \mathcal{H} , such that:*

$$\langle \psi(1, 2) | \mathcal{E}_P(1, 2) | \psi(1, 2) \rangle = 1 \quad (3.1)$$

where

$$\mathcal{E}_P(1, 2) = P^{(1)} \otimes [I^{(2)} - P^{(2)}] + [I^{(1)} - P^{(1)}] \otimes P^{(2)} + P^{(1)} \otimes P^{(2)}. \quad (3.2)$$

While the first definition has extended to the case of identical particles the fundamental feature we have recognised holding true for a non-entangled state of two distinguishable particles, the second definition is necessary to make precise the meaning of the statement “*both constituents possess a complete set of properties*” in the considered peculiar situation where it is not possible, both conceptually and practically, to distinguish the two particles. Actually, the condition of Eq. (3.1) gives the probability of finding *at least* one of the two identical particles (of course, one cannot say which one) in the state associated with the one-dimensional projection operator P . Since, as already noticed, any state vector is a simultaneous eigenvector of a complete set of

³As noticed before these statements do not hold true only in the case of two bosons which are associated with the same state vector.

commuting observables, condition of Eq. (3.1) allows to attribute to *at least* one of the particles the complete set of properties (eigenvalues) associated with the considered set of observables.

With the aid of the previous definitions we have been able to prove [5, 6] the following theorems which identify the mathematical form displayed by non-entangled state vectors of two identical particles:

Theorem 3.1 *The identical fermions \mathcal{S}_1 and \mathcal{S}_2 of a composite quantum system $\mathcal{S} = \mathcal{S}_1 + \mathcal{S}_2$ described by the normalized state $|\psi(1, 2)\rangle$ are **non-entangled** if and only if $|\psi(1, 2)\rangle$ is obtained by antisymmetrizing a factorized state.*

Theorem 3.2 *The identical bosons of a composite quantum system $\mathcal{S} = \mathcal{S}_1 + \mathcal{S}_2$ described by the normalized state $|\psi(1, 2)\rangle$ are **non-entangled** if and only if either the state is obtained by symmetrizing a factorized product of two orthogonal states or it is the product of the same state for the two particles.*

These two theorems characterize the cases in which property attribution to identical particles is still possible in spite of the non-factorizable form of their associated state vectors. It is necessary to point out that in the situation described by Theorems 3.1 and 3.2, not only the property attribution is possible but also the peculiar nonlocal correlations between measurement outcomes which are typical of the entangled states do not occur. Accordingly, no Bell's inequality can be violated and no teleportation process can be performed [5, 6] by means of a state where each particle still possess a definite state vector.

4 Another criterion for detecting entanglement

Recently in the scientific literature new criteria for detecting entanglement appeared [8, 9, 10, 11]. Part of them simply consists in a (careless) extension to the case of identical particles of the same criteria used when dealing with distinguishable particles. Unfortunately those criteria presents some obscure aspects and sometimes they fail to identify certain kinds of non-entangled states: in fact while some of them [8, 9] correctly deal with the case of identical fermions, an inappropriate treatment of the (subtle) boson case is presented in Ref. [9] and in Ref. [10] the use of the entropy criterion is misleading. With the aim of overcoming such puzzling situations, we have presented an unambiguous criterion to identify whether a state is entangled or not, which resort simultaneously *both* to the consideration of the Slater-Schmidt number of the fermionic and bosonic analog of the Schmidt decomposition *and* to the evaluation of the von Neumann entropy of the one-particle reduced statistical operators.

Such a criterion completely agrees with the criterion based on the property attribution which we reviewed in the previous section and it seems to settle all the puzzling issues which have been pointed out by the authors of Refs. [9, 10, 11].

We present such a criterion dealing first with the simpler case of two identical fermions and, subsequently, we pass to analyze the more subtle case of two identical bosons.

4.1 The fermion case

The notion of entanglement for systems composed of two identical fermions has been discussed in Ref. [8] where a “*fermionic analog of the Schmidt decomposition*” was exhibited. This de-

composition derives from an extension to the set of the antisymmetric complex matrices of a well-known theorem holding for antisymmetric real matrices and it states that:

Theorem 4.1.1 *Any state vector $|\psi(1,2)\rangle$ describing two identical fermions of spin s and, consequently, belonging to the antisymmetric manifold $\mathcal{A}(\mathbb{C}^{2s+1} \otimes \mathbb{C}^{2s+1})$, can be written as:*

$$|\psi(1,2)\rangle = \sum_{i=1}^{(2s+1)/2} a_i \cdot \frac{1}{\sqrt{2}} [|2i-1\rangle_1 \otimes |2i\rangle_2 - |2i\rangle_1 \otimes |2i-1\rangle_2], \quad (4.1)$$

where the states $\{ |2i-1\rangle, |2i\rangle \}$ with $i = 1 \dots (2s+1)/2$ constitute an orthonormal basis of \mathbb{C}^{2s+1} , and the complex coefficients a_i (some of which may vanish) satisfy the normalization condition $\sum_i |a_i|^2 = 1$.

The number of non-zero coefficients involved in the decomposition of Eq. (4.1) is called the *Slater number* of the state $|\psi(1,2)\rangle$. We distinguish two cases:

Slater Number = 1. In this situation the state $|\psi(1,2)\rangle$ has the form of a single Slater determinant:

$$|\psi(1,2)\rangle = \frac{1}{\sqrt{2}} [|1\rangle_1 \otimes |2\rangle_2 - |2\rangle_1 \otimes |1\rangle_2] \quad (4.2)$$

Since the state has been obtained by antisymmetrizing the product of two orthogonal states, $|1\rangle$ and $|2\rangle$, it must be considered as non-entangled according to Theorem 3.1. The reduced single-particle statistical operators of each particle (it does not really matter which one we consider since, due to symmetry considerations, they are equal) and their von Neumann entropy (expressed in base 2) are:

$$\rho^{(1 \text{ or } 2)} = \frac{1}{2} [|1\rangle\langle 1| + |2\rangle\langle 2|] \quad (4.3)$$

$$S(\rho^{(1 \text{ or } 2)}) \equiv -\text{Tr} [\rho^{(1 \text{ or } 2)} \log \rho^{(1 \text{ or } 2)}] = 1 \quad (4.4)$$

In this situation, the value $S(\rho^{(1 \text{ or } 2)}) = 1$ correctly measures only the unavoidable uncertainty concerning the quantum state to attribute to each of the two identical physical subsystems and it has nothing to do with any uncertainty arising from any actual form of entanglement. In fact it should be obvious that we cannot pretend that the operator $\rho^{(1 \text{ or } 2)}$ of Eq. (4.3) describes the properties of *precisely* the first or of the second particle of the system, due to their indistinguishability.

Slater number > 1. In this case the state $|\psi(1,2)\rangle$ cannot be obtained by antisymmetrizing the tensor product of two orthogonal states and, consequently, its decomposition involves more than one single Slater determinant. Therefore, as a consequence of the criterion expressed by Theorem 3.1, the state $|\psi(1,2)\rangle$ must be considered as a truly entangled state. The reduced single-particle statistical operators and their associated von Neumann entropy are:

$$\rho^{(1 \text{ or } 2)} = \sum_{i=1}^{(2s+1)/2} \frac{|a_i|^2}{2} [|2i-1\rangle\langle 2i-1| + |2i\rangle\langle 2i|] \quad (4.5)$$

$$S(\rho^{(1 \text{ or } 2)}) = - \sum_{i=1}^{(2s+1)/2} |a_i|^2 \log \frac{|a_i|^2}{2} = 1 - \sum_{i=1}^{(2s+1)/2} |a_i|^2 \log |a_i|^2 > 1 \quad (4.6)$$

In this case the von Neumann entropy is strictly greater than 1 and it correctly measures both the uncertainty deriving from the indistinguishability of the particles and the one connected with the genuine entanglement of the state.

The previous two cases can be summarized in the following theorem:

Theorem 4.1.2 *A state vector $|\psi(1,2)\rangle$ describing two identical fermions is **non-entangled** if and only if its Slater number is equal to 1 or equivalently if and only if the von Neumann entropy of the one-particle reduced statistical operator $S(\rho^{(1 \text{ or } 2)})$ is equal to 1.*

4.2 The boson case

Let us pass now to analyze the case of bipartite quantum systems composed of two identical bosons. This case turns out to be slightly more articulated and subtle than the fermionic case, as a consequence of the peculiar properties of the bosonic statistics. We begin by considering the bosonic Schmidt decomposition of an arbitrary state vector $|\psi(1,2)\rangle$ belonging to the symmetric manifold $\mathcal{S}(\mathbb{C}^{2s+1} \otimes \mathbb{C}^{2s+1})$ and describing two identical bosons:

Theorem 4.2.1 *Any state vector describing two identical s -spin boson particles $|\psi(1,2)\rangle$ and, consequently, belonging to the symmetric manifold $\mathcal{S}(\mathbb{C}^{2s+1} \otimes \mathbb{C}^{2s+1})$ can be written as ⁴:*

$$|\psi(1,2)\rangle = \sum_{i=1}^{2s+1} b_i |i\rangle_1 \otimes |i\rangle_2, \quad (4.7)$$

where the states $\{|i\rangle\}$, with $i = 1, \dots, 2s+1$, constitute an orthonormal basis for \mathbb{C}^{2s+1} , and the real nonnegative coefficients b_i satisfy the normalization condition $\sum_i b_i^2 = 1$.

The number of non-zero coefficients b_i appearing in the decomposition of Eq. (4.7) is called the Schmidt number of the state $|\psi(1,2)\rangle$. Then the following cases can occur:

Schmidt number = 1. In this case the state is factorized, i.e. $|\psi(1,2)\rangle = |i^*\rangle \otimes |i^*\rangle$, and it describes two identical bosons in the same state $|i^*\rangle$. It is evident that such a state must be considered as non-entangled since one knows precisely the properties objectively possessed by each particle and, consequently, there is no uncertainty about which particle has which property. This fact perfectly agrees with the von Neumann entropy of the single-particle reduced statistical operators $S(\rho^{(1 \text{ or } 2)})$ being null.

Schmidt number = 2. According to Eq. (4.7), the most general state with Schmidt number equal to 2 has the following form:

$$|\psi(1,2)\rangle = b_1 |1\rangle_1 \otimes |1\rangle_2 + b_2 |2\rangle_1 \otimes |2\rangle_2, \quad (4.8)$$

where $b_1^2 + b_2^2 = 1$.

⁴It is worth pointing out that the Schmidt decomposition of Eq. (4.7) is not always unique, as happens for the biorthonormal decomposition of states describing distinguishable particles. However, the number of non-zero coefficients is uniquely determined.

Now two subcases, depending on the values of the positive coefficients b_1 and b_2 , must be distinguished and separately analyzed. If they are equal, that is, if $b_1 = b_2 = 1/\sqrt{2}$, the following theorem holds:

Theorem 4.2.2 *The condition $b_1 = b_2 = 1/\sqrt{2}$ is necessary and sufficient in order that the state $|\psi(1,2)\rangle = b_1|1\rangle_1 \otimes |1\rangle_2 + b_2|2\rangle_1 \otimes |2\rangle_2$ can be obtained by symmetrizing the factorized product of two orthogonal states.*

In this situation, and in full accordance with Theorem 3.2, one must consider this state as non-entangled since it is possible to attribute definite state vectors (and consequently definite objective properties) to both particles. As usual, we cannot say which particle is associated with which state due to their indistinguishability. Moreover the von Neumann entropy of the reduced statistical operators $S(\rho^{(1 \text{ or } 2)})$ is equal to 1 and it measures only the uncertainty descending from the indistinguishability of the particles, as happened in the fermion case with the state of Eq. (4.2),

On the contrary, when the two coefficients are different, that is, when $b_1 \neq b_2$, the following Theorem holds:

Theorem 4.2.3 *The condition $b_1 \neq b_2$ is necessary and sufficient in order that the state $|\psi(1,2)\rangle = b_1|1\rangle_1 \otimes |1\rangle_2 + b_2|2\rangle_1 \otimes |2\rangle_2$ can be obtained by symmetrizing the factorized product of two non-orthogonal states.*

According to our original criterion, this state must be considered as a truly entangled state since it is impossible to attribute to both particles definite state vectors (and, consequently, definite objective properties). In this situation the von Neumann entropy of the reduced statistical operator $S(\rho^{(1 \text{ or } 2)}) = -b_1^2 \log b_1^2 - b_2^2 \log b_2^2$ lies within the open interval $(0, 1)$. It correctly measures simultaneously the uncertainty arising both from the indistinguishability of the particles and from the entanglement. It is strictly less than 1 because, in a measurement process, there is a probability greater than $1/2$ to find both bosons in the same physical state ($|1\rangle$ or $|2\rangle$ depending whether $b_1 > b_2$ or vice versa).

Schmidt number ≥ 3 . In this situation the state is a genuine entangled one since it cannot be obtained by symmetrizing a factorized product of two orthogonal states ⁵ and the von Neumann entropy of the reduced statistical operators is such that $S(\rho^{(1 \text{ or } 2)}) \in (0, \log(2s + 1)]$.

As a consequence of our previous analysis, we can exhibit a unified criterion for detecting the entanglement in the boson case. In order to be unambiguous, such a criterion should make simultaneous use of both the Schmidt number and the von Neumann entropy criteria. In fact, as we have seen before, there exist states with Schmidt number equal to 2, or with von Neumann entropy equal to 1, which can be non-entangled as well as entangled. Therefore, the only consistent way to overcome this problem derives from considering the two criteria together, as clearly stated in the next theorem:

⁵In fact, if this would be true, the rank of the reduced density operator would be equal to 2, in contradiction with the fact that a Schmidt number greater than or equal to 3 implies a rank equal to or greater than 3.

Theorem 4.2.4 *A state vector $|\psi(1,2)\rangle$ describing two identical bosons is **non-entangled** if and only if either its Schmidt number is equal to 1, or the Schmidt number is equal to 2 and the von Neumann entropy of the one-particle reduced density operator $S(\rho^{(1\text{ or }2)})$ is equal to 1. Alternatively, one might say that the state is **non-entangled** if and only if either its von Neumann entropy is equal to 0, or it is equal to 1 and the Schmidt number is equal to 2.*

5 Conclusions

The aim of this paper was that of reviewing the delicate problem of deciding whether a state describing a system of two identical particles is entangled or not. Following two different, but totally equivalent approaches, we have presented two criteria which, in our opinion, should have clarified this issue. The first [5, 6], in the spirit of the founder fathers of Quantum Mechanics, is based on the possibility of attributing a complete set of objective properties to both constituents (that is, a definite state vector) while the second [7] is based on the consideration of *both* the Slater-Schmidt number of the fermionic and bosonic analog of the Schmidt decompositions of the states describing the system *and* of the von Neumann entropy of the reduced statistical operators.

References

- [1] E.Schrödinger, *Proceedings of the Cambridge Philosophical Society* **31**, 555 (1935); **32**, 446 (1936).
- [2] C.H.Bennett, G.Brassard, C.Crepeau, R.Josza, A.Peres and W.K.Wootters, *Phys. Rev. Lett.* **70**, 1895 (1993); D.Bouwmeester, J-W.Pan, K.Mattle, M.Eible, H.Weinfurter and A.Zeilinger; *Nature* **390**, 575 (1997).
- [3] P.Shor, *Proc. of 35th Annual Symposium on the Foundations of Computer Science*, IEEE Computer Society, Los Alamitos, 124 (1994); *S.I.A.M. Journal of Computing* **26**, 1484 (1997); L.Grover, *Proc. 28 annual ACM Symposium on the Theory of Computing*, ACM Press New York, 212 (1996).
- [4] A.K.Ekert, *Phys. Rev. Lett* **67**, 661 (1991).
- [5] G.C.Ghirardi, L.Marinatto and T.Weber, *J. Stat. Phys.* **108**, 49 (2002).
- [6] G.C.Ghirardi and L.Marinatto, *Fortschr. Phys.*, **51**, No.4-5, 379 (2003).
- [7] G.C.Ghirardi and L.Marinatto, to appear on *Phys. Rev. A* **70**, (2004).
- [8] J.Schliemann, J.I.Cirac, M.Kus, M.Lewenstein, and D.Loss, *Phys. Rev. A* **64**, 022303 (2001).
- [9] R.Paskauskas and L.You, *Phys. Rev. A* **64**, 042310 (2001).
- [10] Y.S.Li, B.Zeng, X.S.Liu and G.L.Long, *Phys. Rev. A* **64**, 054302 (2001).
- [11] H.M.Wiseman and J.A.Vaccaro, *Phys. Rev. Lett.* **91**, 097902 (2003).